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Reliability of Computational Analysis of Plasticity Problems

by

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The paper addresses the reliability of computational analysis of plasticity problems in light of the experimental work and its mathematical formulation. The paper also addresses some basic constitutive laws used in engineering and proposes one which leads to a well posed mathematical problem and agrees well with the experimental data. The data of the experimental project for 5454 aluminum alloy in the H32 condition for periodic and random strains are presented. Various constitutive laws are assessed in light of the experimental data. Outstanding problems related to the reliability of the analysis are mentioned.

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Reliability of Computational Analysis of Plasticity Problems

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Dedicated to Professor E. Stein
on the occasion of his 60th birthday

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Abstract. The paper addresses the reliability of computational analysis of plasticity problems in light of the experimental work and its mathematical formulation. The paper also addresses some basic constitutive laws used in engineering and proposes one which leads to a well posed mathematical problem and agrees well with the experimental data. The data of the experimental project for 5454 aluminum alloy in the H32 condition for periodic and random strains are presented. Various constitutive laws are assessed in light of the experimental data. Outstanding problems related to the reliability of the analysis are mentioned.

1) Introduction

The problem of the reliability of the computational analysis of solid mechanics relies on the assumptions made. In this paper we will discuss the question of nonlinear constitutive laws especially related to the plasticity problem.

The study of plasticity and cyclic loading has generated a large amount of literature. We refer here for example to [1] and an extensive list of references there. We will be especially concerned about proposed models describing plastic strain hardening during general cyclic loading (e.g., of random character). These models are either empirical or are related to some mechanical ideas. As examples we refer to [1-7]. The models used in practice are usually one dimensional, are tested in very limited sets of experimental data and are extended to higher dimensional problems by some mechanical hypotheses (e.g., v. Mises, Tresca, Hencky, etc.).

In this paper we will address the problem of the selection of the constitutive law especially in one dimension. Also, the generalization to higher dimension will be mentioned briefly.

2) The one-dimensional model problem

The typical quasistatic model problem (in one-dimension) in solid mechanics is the following: Let $I = [0,1]$, $W = [0,\infty)$ and $f(x,t)$, $f(x,0) = 0$ piecewise analytic in t and $L_2(I)$ for every t, be given. The problem is to find $u(x,t)$ and $\sigma(x,t)$, $(x,t) \in I \times W$ such that

$$\frac{\partial\sigma}{\partial x}(x,t) = f(x,t) \tag{2.1}$$

$$\frac{\partial u}{\partial x}(x,t) = \varepsilon(x,t) \tag{2.2}$$

$$\dot{\sigma}(x,t) = A(\dot{\varepsilon}(x,t)) \tag{2.3}$$

$$u(0,t) = u(1,t) = 0 \tag{2.4}$$

$$u(x,0) = 0 \tag{2.5}$$

where

$$\dot{\sigma}(x,t) = \frac{\partial\sigma}{\partial t}(x,t)$$

$$\dot{\varepsilon}(x,t) = \frac{\partial\varepsilon}{\partial t}(x,t)$$

and A is an operator which maps any function $e(t)$ (strain) continuous on $[0,\infty)$, $e(0) = 0$ into the function $s(t)$ (stress) which is continuous on $[0,\infty)$ with $s(0) = 0$ and $s(t_0)$, $t_0 > 0$ depends only on $e(t)$ for $0 \le t \le t_0$. (Hence we assume that A in (2.3) is x-independent). Operator A is the *constitutive law operator*. One of the main problems is to have an operator which *guarantees* reasonable properties of the solution of (2.1)-(2.5) such as the existence and uniqueness. In addition the operator has to approximate the material behavior well.

Let us now formulate a specific class of constitutive operators A which describe the plastic behavior and guarantee that the problem (2.1)-(2.5) is a well posed problem. We will introduce the family of the constitutive laws which are based on two following basic assumptions:

1) Existence of a convex yield surface.

2) The normality condition: the plastic increment is proportional to the outward normal to the yield surface during plastic flow.

Let $\mathfrak{U}$ be a convex set in $\mathbb{R}^n$, and let $F(s,\alpha)$ be defined on $\mathbb{R}\times\mathfrak{U}$ so that

(2.6) $F(s,\alpha)$ is convex and C^1 functional on $\mathbb{R}\times\mathfrak{U}$.

(2.7) $$F(0,0) = 0.$$

(2.8) There exist constants γ, Γ such that $0 < \gamma \le \left|\frac{\partial F}{\partial s}\right|, \left|\frac{\partial F}{\partial \alpha}\right| \le \Gamma$ uniformly on the set $\{(x,\alpha)|F(s,\alpha) = Z_0\}$ for some $Z_0 > 0$.

The function $F(s,\alpha) = Z_0$ is the yield surface. Let us now derive the constitutive law using the above two postulates.

Let (s,α) lie inside the yield surface, i.e., $F(s,\alpha) < Z_0$. Then the material will be assumed elastic

(2.9a) $$\dot{s} = E\dot{e}$$

(2.9b) $$\dot{\alpha} = 0.$$

In (2.9a) E denotes the model of elasticity.

If (s,α) lies on the yield surface, i.e., $F(s,\alpha) = Z_0$ then we will split the strain increment $\dot{e}$ into elastic and plastic parts

(2.10) $$\dot{e} = \dot{e}^E + \dot{e}^P$$

and

(2.11) $$\dot{s} = E\dot{\varepsilon}^E = E(\dot{\varepsilon}-\dot{\varepsilon}^P)$$

Using the normality condition we get the following equation

(2.11a) $$\dot{s} = \left\{E - \frac{E^2\left(\frac{\partial F}{\partial s}\right)^2}{\left(\frac{\partial F}{\partial \alpha}\right)^T\left(\frac{\partial F}{\partial \alpha}\right) + E\left(\frac{\partial F}{\partial s}\right)^2}\right\}\dot{\varepsilon}$$

(2.11b)
$$\dot{\alpha} = -\frac{\frac{\partial F}{\partial \alpha}}{\left[\frac{\partial F}{\partial \alpha}\right]^T \left[\frac{\partial F}{\partial \alpha}\right]} \left[\frac{\partial F}{\partial s} \dot{s}\right]$$

where $\left[\frac{\partial F}{\partial \alpha}\right]^T = \left[\frac{\partial F}{\partial \alpha_1}, \cdots, \frac{\partial F}{\partial \alpha_m}\right]$.

Hence, the constitutive law under consideration is

(2.12a), (2.12b)
$$\left.\begin{aligned} \dot{s} &= E\dot{\varepsilon} \\ \dot{\alpha} &= 0 \end{aligned}\right\} \quad \text{for} \quad (s,\alpha) \in \mathcal{E}$$

(2.13a), (2.13b)
$$\left.\begin{aligned} \dot{s} &= \left[E - \frac{E^2\left[\frac{\partial F}{\partial s}\right]^2}{\left[\frac{\partial F}{\partial \alpha}\right]^T \frac{\partial F}{\partial \alpha} + E\left[\frac{\partial F}{\partial s}\right]^2}\right] \dot{e} \\ \dot{\alpha} &= -\frac{\left[\frac{\partial F}{\partial \alpha}\right]}{\left[\frac{\partial F}{\partial \alpha}\right]^T \left[\frac{\partial F}{\partial \alpha}\right]} \left[\frac{\partial F}{\partial s} \dot{s}\right] \end{aligned}\right\} \quad \text{for} \quad (s,\alpha) \in \mathcal{P}$$

where

(2.14a)
$$\mathcal{E} = \left\{(s,\alpha) \,\middle|\, F(s,\alpha) < Z_0 \quad \text{or} \quad F(s,\alpha) = Z_0 \quad \text{and} \quad \frac{\partial F}{\partial s}\dot{s} \le 0\right\}$$

(2.14b)
$$\mathcal{P} = \left\{(s,\alpha) \,\middle|\, F(s,\alpha) = Z_0 \quad \text{and} \quad \frac{\partial F}{\partial s}\dot{s} > 0\right\}$$

Obviously, operator A is now defined by (2.13) and (2.14) for all $e(t)$ which are continuous and piecewise differentiable. (We assume $e(0) = \alpha(0) = s(0) = 0$).

The family of the constitutive laws (2.12)-(2.14) based on the yield surface $F(s,\alpha)$ satisfying (2.6), (2.8) will be called an *admissible plastic law* or briefly an *admissible law*.

Now we have

Theorem 2.1. Let A be admissible. Then the problem (2.1) has unique solution with $\varepsilon(x,t)$, $\sigma(x,t)$ piecewise analytic in t for every x and $L^2(I)$ for every t.

Theorem 2.2. Let A be admissible. Then for every mesh and linear element the finite element solution of the problem (2.1) exists is unique and is piecewise analytic in t. Moreover, if we let the mesh size tend to zero, then the sequence of the finite element solution will weakly* converge to the solution of the problem (2.1).

Theorems 2.1 and 2.2 have been proven in a general form in two dimensions (under weaker conditions than we mentioned) in [8]. See also [9].

3) One-dimensional engineering constitutive laws

In engineering the constitutive laws of plasticity are formulated in different ways, usually in an incremental form using the notion of reversals. Nevertheless these laws can be cast in a form of differential equation. This form will be called the *standard form.*

Let us mention some of these laws.

a) *Bilinear kinematic hardening law.* Here we have one internal parameter $(n = 1)$ and

$$\begin{aligned} &(3.1a) \quad \dot{s} = E\dot{e} \\ &(3.1b) \quad \dot{\alpha} = 0 \end{aligned} \quad \text{for} \begin{cases} |s-k^{-1}\alpha| < s_0 \\ \text{or} \quad s - k^{-1}\alpha = s_0 \text{ and } \dot{s} \le 0 \\ \text{or} \quad -s + k^{-1}\alpha = s_0 \text{ and } \dot{s} \ge 0 \end{cases}$$

$$\begin{aligned} &(3.2a) \quad \dot{s} = E_p\dot{e} \\ &(3.2b) \quad \dot{\alpha} = E_p k\dot{e} \end{aligned} \quad \text{for} \begin{cases} s - k^{-1}\alpha = s_0 \text{ and } \dot{s} > 0 \\ -s + k^{-1}\alpha = s_0 \text{ and } \dot{s} < 0 \end{cases}$$

The operator A based on (3.1) and (3.2) is independent of k. It depends on the model of elasticity E, the model of plastic flow E_p and the yield stress s_0. Of course $\alpha(t)$ depends on k. Usually $k = 1$ is used. Then, in a physical interpretation, this law keeps the shape and size of the yield surface fixed, and the surface only shifts by the hardening. The parameter α is the position of the center of the yield surface.

The problem is whether we can cast (3.1), (3.2) in a form (2.12)-(2.14) with $F(s,\alpha)$ satisfying (2.6)-(2.8). Selecting $k = \left(\frac{E-E_p}{EE_p}\right)^{1/2}$, (3.1), (3.2) will coincide with (2.12)-(2.14) with

(3.3) $$F(s,\alpha) = [\max F_1(s,\alpha), F_2(s,\alpha)]^*$$

where $[\cdot,\cdot]^*$ means a smoothening operator in the neighborhood $Q = \{(s,\alpha) | F_1(s,\alpha) = F_2(s,\alpha)\}$. (This smoothening is only formal to achieve C^1 continuity of $F(s,\alpha)$). In (3.3) we have

(3.4a) $$F_1(s,\alpha) = s - \left(\frac{EE_p}{E-E_p}\right)^{1/2} \alpha$$

(3.4b) $$F_2(s,\alpha) = -s + \left(\frac{EE_p}{E-E_p}\right)^{1/2} \alpha.$$

We mention that the value $k^2 = (E-E_p)/EE_p$ is the only value which allows to cast (3.1), (3.2) into the form (2.12)-(2.14).

b) *Bilinear isotropic hardening law.* Here the standard form is as follows

(3.5a) $$\dot{s} = E\dot{e}$$

(3.5b) $$\dot{\alpha} = 0$$

$$\text{for} \begin{cases} & |s| - k^{-1}\alpha < s_0 \\ \text{or} & s - k^{-1}\alpha = s_0 \text{ and } \dot{s} \le 0 \\ \text{or} & -s - k^{-1}\alpha = s_0 \text{ and } \dot{s} \ge 0 \end{cases}$$

$$\dot{s} = E_p \dot{e} \qquad (3.6a)$$

$$\dot{\alpha} = E_p k \dot{e} \qquad (3.6b)$$

for $s - k^{-1}\alpha = s_0$ and $\dot{s} > 0$

$$\dot{s} = E_p \dot{e} \qquad (3.7a)$$

$$\dot{\alpha} = E_p k \dot{e} \qquad (3.7b)$$

for $\left\{ -s - k^{-1}\alpha = s_0 \right.$ and $\dot{s} < 0.$

Once more, the operator A here does not depend on k, while $\alpha(t)$ does.

When $k = 1$ then an analogous physical interpretation as before holds. The yield function keeps shape and its center fixed and the hardening increases its size. Once more the proper selection of k, namely $k = \left(\frac{E-E_p}{EE_p}\right)^{1/2}$ allows to cast this law in the form (2.12), (2.14) with

$$\hat{F}(s,\alpha) = [\max \hat{F}_1(s,\alpha), \hat{F}_2(s,\alpha)]^* \qquad (3.8)$$

$$\hat{F}_1(s,\alpha) = s - \left(\frac{E-E_p}{E-E_p}\right)^{1/2} \alpha \qquad (3.9a)$$

$$F_2(s,\alpha) = -s - \left(\frac{E-E_p}{E-E_p}\right)^{1/2} \alpha. \qquad (3.9b)$$

We get the operator A which coincides with the one defined by (3.5)-(3.7).

Functions $F(s,\alpha)$ and $\hat{F}(s,\alpha)$ satisfy the condition (2.6)-(2.8).

c) *Chaboche law [2].* This law has become recently popular and (see Sect. 4) approximates the experimental results relatively well. Although it is formulated in [2] in a different form, it is possible to write it in the standard form.

$$
\left.
\begin{aligned}
&(3.10a) \qquad \dot{s} = E\dot{e} \\
&(3.10b) \qquad \dot{\chi} = 0 \\
&(3.10c) \qquad \dot{R} = 0 \\
&(3.10d) \qquad h = 0 \\
&(3.10e) \qquad \ell = 0
\end{aligned}
\right\}
\quad \text{for} \quad
\begin{cases}
e_\ell < e < e_h & \text{or} \\
e = e_h \ \text{and} \ \dot{e} \le 0 \\
e = e_\ell \ \text{and} \ \dot{e} \ge 0
\end{cases}
$$

$$
\left.
\begin{aligned}
&(3.11a) \qquad \dot{s} = \frac{E[c(a-\chi) + b(Q-R)]}{c(a-\chi) + b(Q-R) + E}\,\dot{e} \\
&(3.11b) \qquad \dot{\chi} = \frac{Ec(a-\chi)}{c(a-\chi) + b(Q-R) + E}\,\dot{e} \\
&(3.11c) \qquad \dot{R} = \frac{Eb(Q-R)}{c(a-\chi) + b(Q-R) + E}\,\dot{e} \\
&(3.11d) \qquad \dot{e}_h = \dot{e} \\
&(3.11e) \qquad \dot{e}_\ell = \dot{e} - 2\,\frac{R}{E}
\end{aligned}
\right\}
\quad \text{for} \quad
\begin{cases}
e_h = e \\
\text{and} \\
\dot{e} > 0
\end{cases}
$$

$$
\left.
\begin{aligned}
&(3.12a) \qquad \dot{s} = \frac{E[c(a+\chi) + b(Q-R)]}{c(a+\chi) + b(Q-R) + E}\,\dot{e} \\
&(3.12b) \qquad \dot{\chi} = \frac{Ec(a+\chi)}{c(a+\chi) + b(Q-R) + E}\,\dot{e} \\
&(3.12c) \qquad \dot{R} = \frac{-Eb(Q-R)}{c(a+\chi) + b(Q-R) + E}\,\dot{e} \\
&(3.12d) \qquad \dot{e}_h = \dot{e} + 2\,\frac{R}{E} \\
&(3.12e) \qquad \dot{e}_\ell = \dot{e}
\end{aligned}
\right\}
\quad \text{for} \quad
\begin{cases}
e = e_\ell \\
\text{and} \\
\dot{e} < 0
\end{cases}
$$

with the initial conditions $s(0) = \chi(0) = R(0) = 0$,

$$e_h(0) = e_0\,, \qquad e_\ell(0) = -e_0 \tag{3.13}$$

In [2] the law is derived using certain physical heuristic arguments. The following coefficients are input in the Chaboche law.

a: kinematic coefficient

c: kinematic exponents

Q: isotropic coefficient

b: isotropic exponent

e_0: yield strain

E: elastic modulus

In contrast to the kinematic and isotropic laws mentioned earlier, the Chaboche law cannot be cast to the form of an admissible law as simple as before. Nevertheless, an admissible law which is close to the Chaboche law can be formulated. In [10] such law (with n = 2) was proposed analyzed in detail.

d) *The Babuska-Li law*

We have

$$\left.\begin{array}{lll} (3.14a) & \dot{s} = E\dot{e} & \\ (3.14b) & \dot{\alpha} = 0 & \text{for} \\ (3.14c) & \dot{\beta} = 0 & \end{array}\right\} \quad (s,\alpha,\beta) \in \mathcal{E}$$

$$\left.\begin{aligned}
&(3.15a) \qquad \dot{s} = \frac{E\left[\left(\frac{c}{2}\alpha - \sqrt{ac}\right)^2 + \left(\frac{b}{2}\beta - \sqrt{bQ}\right)^2\right]}{\left(\frac{c}{2}\alpha - \sqrt{ac}\right)^2 + \left(\frac{b}{2}\beta - \sqrt{bQ}\right)^2 + E}\,\dot{e} \\
&(3.15b) \qquad \dot{\alpha} = \frac{-E\left(\frac{c}{2}\alpha - \sqrt{ac}\right)}{\left(\frac{c}{2}\alpha - \sqrt{ac}\right)^2 + \left(\frac{b}{2}\beta - \sqrt{bQ}\right)^2 + E}\,\dot{e} \qquad \text{for} \\
&(3.15c) \qquad \dot{\beta} = \frac{-E\left(\frac{b}{2}\beta - \sqrt{bQ}\right)}{\left(\frac{c}{2}\alpha - \sqrt{ac}\right)^2 + \left(\frac{b}{2}\beta - \sqrt{bQ}\right)^2 + E}\,\dot{e}
\end{aligned}\right\} \quad (s,\alpha,\beta) \in \mathcal{P}_+$$

$$\left.\begin{aligned}
&(3.16a) \qquad \dot{s} = \frac{E\left[\left(\frac{c}{2}\alpha + \sqrt{ac}\right)^2 + \left(\frac{b}{2}\beta - \sqrt{bQ}\right)^2\right]}{\left(\frac{c}{2}\alpha + \sqrt{ac}\right)^2 + \left(\frac{b}{2}\beta - \sqrt{bQ}\right)^2 + E}\,\dot{e} \\
&(3.16b) \qquad \dot{\alpha} = \frac{E\left(\frac{c}{2}\alpha + \sqrt{ac}\right)}{\left(\frac{c}{2}\alpha + \sqrt{ac}\right)^2 + \left(\frac{b}{2}\beta - \sqrt{bQ}\right)^2 + E}\,\dot{e} \qquad \text{for} \\
&(3.16c) \qquad \dot{\beta} = \frac{E\left(\frac{b}{2}\beta - \sqrt{bQ}\right)}{\left(\frac{c}{2}\alpha + \sqrt{ac}\right)^2 + \left(\frac{b}{2}\beta - \sqrt{bQ}\right)^2 + E}\,\dot{e}
\end{aligned}\right\} \quad (s,\alpha,\beta) \in \mathcal{P}_-$$

where

$$\begin{aligned}
&(3.17a) \qquad \mathcal{E} = \Big\{(s,\alpha,\beta) \mid F(s,\alpha,\beta) < Z_0 \quad \text{or} \\
&\qquad\qquad F_1(s,\alpha,\beta) = Z_0 \quad \text{and} \quad \dot{e} \le 0 \quad \text{or} \\
&\qquad\qquad F_2(s,\alpha,\beta) = Z_0 \quad \text{and} \quad \dot{e} \ge 0\Big\}
\end{aligned}$$

(3.17b) $$\mathcal{P}_+ = \left\{(s,\alpha,\beta) \mid F_1(s,\alpha,\beta) = Z_0 \text{ and } \dot{e} > 0\right\}$$

(3.17c) $$\mathcal{P}_- = \left\{(s,\alpha,\beta) \mid F_2(s,\alpha,\beta) = Z_0 \text{ and } \dot{e} < 0\right\}$$

and $Z_0 = s_y$ is initial yield stress.

Here

(3.18a) $$F_1(s,\alpha,\beta) = \frac{c}{4}\alpha^2 - \sqrt{ac}\,\alpha + \frac{b}{4}\beta^2 - \sqrt{bQ}\,\beta + s$$

(3.18b) $$F_2(s,\alpha,\beta) = \frac{c}{4}\alpha^2 + \sqrt{ac}\,\alpha + \frac{b}{4}\beta^2 - \sqrt{bQ}\,\beta - s$$

and

(3.19) $$F(s,\alpha,\beta) = [\max(F_1(s,\alpha,\beta),\ F_2(s,\alpha,\beta)]^*.$$

We see that this law has two internal parameters $(n = 2)$ and depends on the constants a, b, c, Q, E, Z_0.

It is easy to see that F defined by (3.19) satisfies (2.6) and (2.7). The property (2.8) is more complicated and needs some relations among coefficients of input data. For the detailed analysis of this law we refer to [10]. In [10] also other laws are proposed and analyzed in detail.

4) Experimental data

It is obvious that the constitutive law influences essentially any computed data (for example, data computed by the finite element method). Hence it is essential to analyze the reliability of the assumed material behavior under realistic conditions. By this, it is meant the behavior of commercial material (without laboratory treatment) under various random strain conditions. The experiments related to the constitutive law have to address the following:

a) Reproducibility of the stress response for the imputed strain.

b) Selection and quantitative assessment of an constitutive law.

For our experimental program we selected 5454 aluminum alloy in the H32 condition. The major alloying elements in the workhardening material are magnesium (2.7%), manganese (0.8%) and chromium (0.12%). This particular alloy is widely used in marine applications and is generally available in a variety of structural shapes as sheet, plate, tube, wire, rod and bar. The normal mechanical properties are tensile strength of 40 ksi, yield strength of 30 ksi and elongation 10%. Samples of the material were obtained from two suppliers.

For the testing program three different classes of strain were used: a triangular (or constant magnitude) strain functions and two different random strain functions (using random number generator for different means and standard deviation). For these cases seven tests were conducted with each test having a constant mean strain level ranging from -0.6% to 0.6%. All tests had a strain range of 1.0% and constant strain rate 0.001/sec. Replicate tests were concluded for each strain, thus total of 84 data sets were created. A typical data set contained approximately 40,000 data points. For two samples of measured data σ the functions φ_i^{σ}, Ψ_{ui}^{σ}

$$\varphi_i^{\sigma} = \sigma_1(t_i) - \sigma_2(t_i)$$

$$\frac{\Psi_i^{\sigma} = \sigma_1(t_i) + \sigma_2(t_i)}{2}$$

were computed together with the norms

$$\|\varphi^{\sigma}\|_{\infty} = \max_{1\le i\le N} |\varphi_i^{\sigma}|$$

$$\|\Psi^{\sigma}\|_{\infty} = \max_{1\le i\le N} |\Psi_i^{\sigma}|$$

As an error measure of the difference the ratio

$$B^{\sigma} = \|\varphi^{\sigma}\|_{\infty} / \|\Psi^{\sigma}\|_{\infty}$$

is used.

For the measure of strain reproducibility the ratio B^{e} was computed (i.e., two identical random strains were compared) and $B^{e} < 1.8\%$ was obtained.

The constants for the constitutive laws were computed for every sample by minimization of the (relative) difference between the experimental and computed data leading to minimal error measure. Typical results are given in tables 1 and 2. Here the results are for 28 periodic and 28 random strains for 10 loading loops. In Table 1 the error measure B^{σ} is reported for the group of periodic and random strain. Here the values of the constants in the constitutive laws are the averages of the best fit constants for every class separately. We report the smallest (best) and largest (worst) error measure of the difference between the samples and the law used as well as the average error measure. In table 2 we report analogous data for the combined set of periodic and random history (56 samples). We use the average constants of the best fits for all (56) samples and the average of the best fit constants for the random strain (28 samples) only. Tables 1 and 2 show clearly that the periodic and random strain lead to different results. They also show the importance of the selection of the values of the constants in the constitutive laws.

Table 3 shows the results for the error measure B^{σ} for one sample (FC), the span of 10, 20, 50 and 100 loops for *the random strain*. The constants in the constitutive laws are the average values of 28 samples of *random strain* and 10 loops. We report the error measure of the difference in stresses between the experiment (sample FC) (mean strain 0.006) and the stresses computed from the law.

Table 1

	28 CYCLIC PERIODIC LOAD HISTORY			28 RANDOM LOAD HISTORY		
	BEST	AVERAGE	WORST	BEST	AVERAGE	WORST
ISOTROPIC	17.5678%	26.1709%	34.8327%	25.9049%	36.6324%	54.3445%
KINEMATIC	13.9703%	21.9894%	29.6227%	24.9118%	30.3313%	37.9495%
CHABOCHE	10.0580%	13.4342%	19.7299%	9.7581%	15.1567%	21.6382%
BABUSKA-LI	8.7208%	13.0558%	17.6285%	11.1595%	17.7304%	23.6274%

Table 2

ALL 28 CYCLIC PERIODIC LOAD AND 28 RANDOM LOAD HISTORY TOGETHER

	MEAN OF 56 REAL DATAS			MEAN OF 28 RANDOM LOAD DATAS		
	BEST	AVERAGE	WORST	BEST	AVERAGE	WORST
ISOTROPIC	20.2992%	32.9285%	55.9337%	20.2992%	36.6324%	32.2566%
KINEMATIC	15.7110%	36.6296%	55.7103%	19.5086%	29.0444%	37.9495%
CHABOCHE	9.1939%	14.3332%	21.9578%	9.0194%	14.2968%	21.6382%
BABUSKA-LI	9.5089%	19.3730%	36.2526%	8.9993%	16.5079%	23.6274%

Table 3

	10 LOOPS	20 LOOPS	50 LOOPS	100 LOOPS
FC - HH	18.6020%	18.1069%	17.6902%	19.0026%
FC - ISOTROPIC	41.9331%	45.4107%	44.1721%	44.1659%
FC - KINEMATIC	27.7171%	33.2344%	32.8032%	32.5333%
FC - CHABOCHE	21.4401%	21.1359%	24.1460%	26.6967%
FC - BABUSKA-LI	21.3331%	21.7366%	26.3657%	24.7862%

Table 4

	10 LOOPS	20 LOOPS	50 LOOPS	100 LOOPS
GL - ES	10.2216%	10.2265%	10.6647%	10.6341%
GL - ISOTROPIC	29.1335%	31.9397%	34.6639%	36.8122%
GL - KINEMATIC	24.5334%	27.4146%	30.2475%	32.4730%
GL - CHABOCHE	12.7273%	12.4393%	11.9824%	11.5788%
GL - BABUSKA-LI	12.3832%	12.0705%	11.5576%	11.1017%

The first row reports the error measure of the stresses for two samples (FC and HH) (for the same random strain). This data characterize the reproducibility of the response and are related to the uncertainty of the material properties.

Table 4 shows analogous results *for periodic strain* (once more the constants in the constitutive laws were computed from the random strain for 10 loops and 28 samples). The error measure for the difference between two samples (GS and ES) (mean strain is -0.004) is reported too. We see once more the difference of the responses for the random and periodic load. Periodic load response is obviously more predictable. This clearly shows the need to make experiments not only for periodic strain.

Figure 1 shows the difference in stresses for the sample FC and different constitutive laws for the random strain. Figure 2 shows analogous results for the periodic strain (for the sample GL). Figure 3 shows the strain-stress relation for the sample FC and random constitutive laws.

We refer to [11] for additional experimental results, statistical analysis of the obtained data and also comparison with other laws. See [11] also for details of the experimental procedures. Obviously unreliability (uncertainty) in the material responses is large and hence reliability (uncertainty) in computed (finite element) data is expected to be large too. We will briefly address this problem in Section 6.

5) Two dimensional constitutive law

The problem is *in principle* the same as in one dimension. Here instead the scalar strain and stress we have to consider strain and stress tensors. Also the internal parameter space is larger.

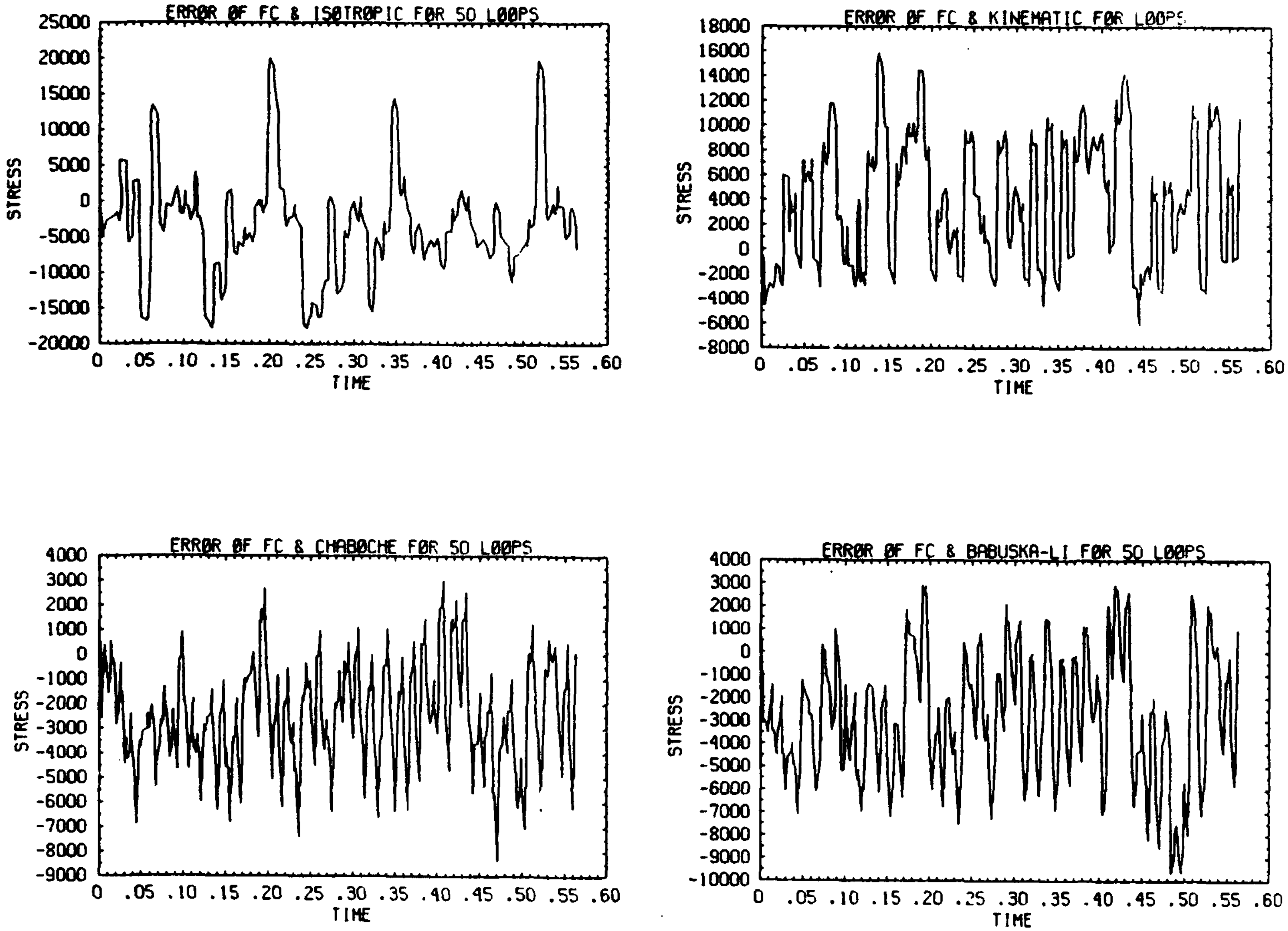

Figure 1. The difference in stresses of the experiment and constitutive law for random strain.

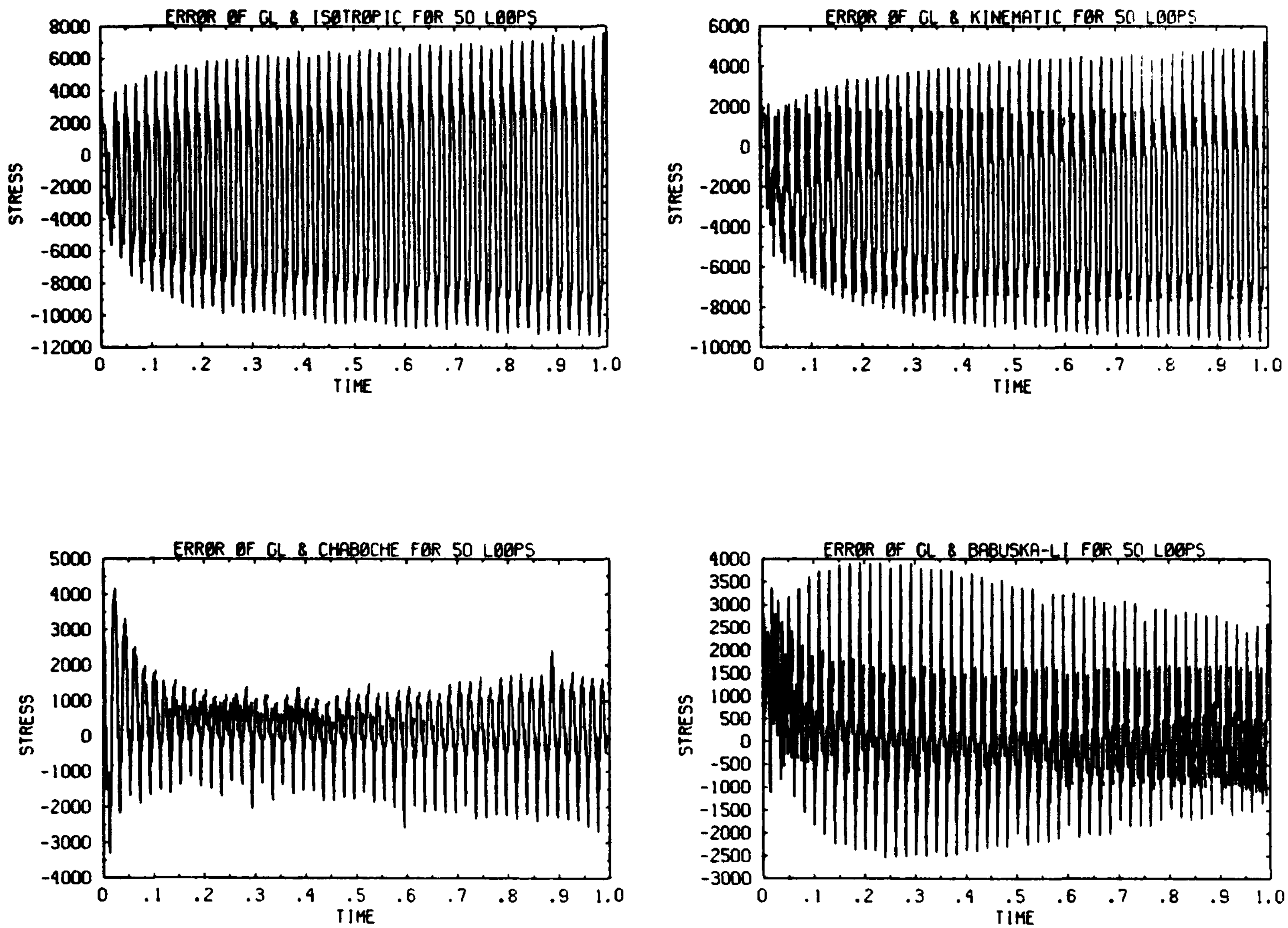

Figure 2. The difference in stresses of the experiment and constitutive law for periodic strain.

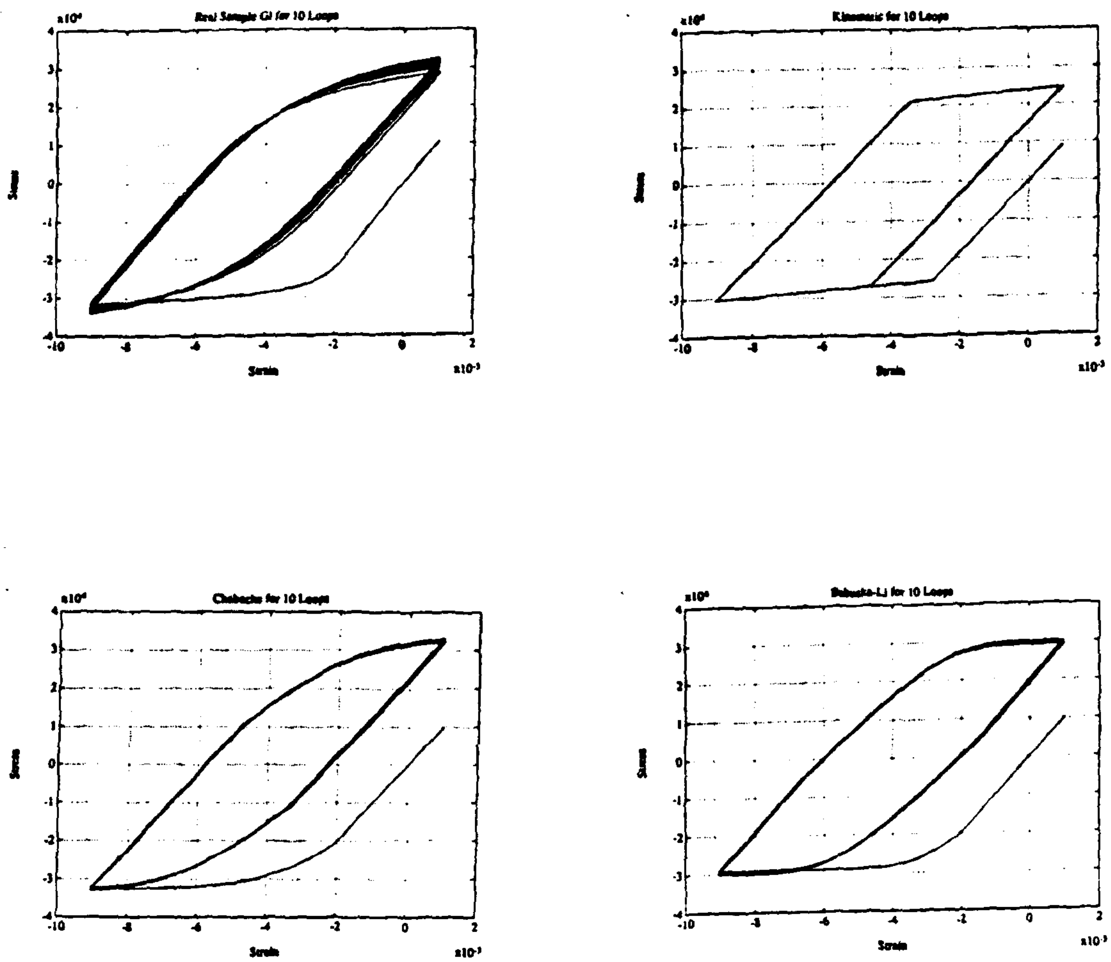

Figure 3. The stress strain relation for the experiment and for various laws and periodic strain.

Because of complexity of the experimentation, one dimensional laws are usually used together with additional hypothesis such as von Mises, etc. In the case of isotropic material we need rotational symmetry of the law. The generalization of Babuska-Li law (see [10] preserving rotational symmetry leads to the following function $F(s,\alpha,\beta)$ where s is tensor with the components $s = (s_x, s_y, s_{xy}) \in \mathbb{R}^3$, $\alpha = [\alpha_1, \alpha_2, \alpha_3] \in \mathbb{R}^3$ and $\beta \in \mathbb{R}^1$.

$$F(s,\alpha,\beta) = f(s - \sqrt{ac}\sqrt{R}\,\alpha) + \frac{c}{4} f^2(\sqrt{R}\,\alpha) + \frac{b}{4}\beta^2 - \sqrt{bQ}\,\beta$$

where
$$f^2(s) = s_x^2 + s_y^2 - s_x s_y + 3s_{xy}^2$$

$$R = \begin{bmatrix} 4/3 & 2/3 & 0 \\ 2/3 & 4/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

In the constitutive law, the strain e has to be in the form $\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}$. For details, analysis, and proof we refer to [10] where additional results are presented.

6) Reliability of the computational analyses of plasticity problems

Reliability of the computational analyses depends on

a) Mathematical modelling restricting the characteristic features of the physical problem, available information and aims of the analysis.

b) Correctness of the mathematical formulation guaranteeing the basic properties as existence of the solution, etc.

c) Numerical treatment which produces an approximate solution which is sufficiently close to the exact solution in a sense associated with the aims of the analyses.

As has been seen in the previous section, an important characteristic of plasticity problem is the large uncertainty in available information, especially in the constitutive law. In 2 and 3 dimensional problems, the uncertainty described by the error measure B is of order 30%-40% or more. (In the *one dimensional* setting we have seen in the best cases uncertainty of order 20-25%.)

The following problems arise:

1) How much the of uncertainty in the constitutive law is projected in the uncertainty of the data of interest?

2) Is it possible to formulate and solve the problem to determine the optimal bounds for the data of interest depending on uncertainties in available information?

3) How can one formulate and sol' ' the problem in a stochastic sense? Here, of course, various probability fields or at least correlations of the experimental material properties would be needed and are practically unavailable today.

In [8] an attempt to derive bounds depending on the uncertainties in the constitutive law of the worst scenario was made. The idea is to assume that all uncertainty in constitutive law is in the unknown past. The problem is then to consider the "worst" possible past (in a set of admissible function) leading to the largest bound of data of interest.

There are other possibilities to address: the influence of the uncertainties analyzed by Monte Carlo methods and approaches of the type used in optimal design, etc.

It seems that any finite element consideration in plasticity problems using constitutive law without addressing the effects of uncertainties has to be interpreted and accepted in a very conservative way.

The problem of the correct mathematical formulation is a crucial one. The formulation of the (direct) plasticity problem guaranteeing well posedness can be achieved without any adverse effects of approximating experimental data as has been seen in Section 4.

The numerical treatment should lead to the approximate solution with acceptable accuracy in the sense needed for application. Today's available weak convergence theory seems to be far from what is needed.

Summarizing we see that there are major problems and questions related to the reliability of the computational analysis of the plasticity problems.

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